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Potential Contribution of CNN-based Solving of Stiff ODEs & PDEs to Enabling Real-Time Computational Engineering

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Abstract — **One of the most common approaches to avoid complexity while numerically solving stiff ordinary differential equations (ODEs) is approximating them by ignoring the nonlinear terms. While facing stiff partial differential equations (PDEs) the same is done by avoiding/suppressing the nonlinear terms from the Taylor's series expansion. By so doing, the traditional methods for solving stiff PDEs and ODEs do compromise on both efficiency and precision of the resulting computations. This does inevitably lead to less accurate results that consequently cannot provide the full insight that may be needed in diverse cutting-edge situations in the 'real' nonlinear dynamical behavior experienced by the various engineering and natural systems (generally modeled by nonlinear differential equations of the types ODE or PDE), which are analyzed in the frame of the novel discipline called Computational Engineering. For many of these systems, even a real-time simulation and/or control of the behavior is wished or needed; this sets evidently extremely high challenging requirements to the computing capability with regard to both computing speed and precision. This paper develops/proposes and validate through a series of presentable examples a comprehensive high-precision and ultra-fast computing concept for solving stiff ODEs and PDEs with Cellular Neural Networks (CNN). The core of this concept is a straight-forward scheme that we call 'Nonlinear Adaptive Optimization (NAOP)', which is used for a precise template calculation for solving any (stiff) nonlinear ODE through CNN processors. One of the key contributions of this work, this is a real breakthrough, is to demonstrate the possibility of mapping/transforming different types of nonlinearities displayed by various classical and well-known oscillators (e.g. van der Pol-, Rayleigh-, Duffing-, Rössler-, Lorenz-, and Jerk- oscillators, just to name a few) unto first-order CNN elementary cells, and thereby enabling the easy derivation of corresponding CNN templates. Furthermore, in case of PDE solving, the same concept also allows a mapping unto first-order CNN cells while considering one or even more nonlinear terms of the Taylor's series expansion generally used in the transformation of a PDE in a set of coupled nonlinear ODEs. Therefore, the concept of this paper does significantly contribute to the consolidation of CNN as a universal and ultra-fast solver of stiff differential equations (both ODEs and ODEs). This clearly enables a CNN-based, real-time, ultra-precise, and low-cost Computational Engineering. As proof of concept some well-known prototypes of stiff equations (van der Pol, Lorenz, and Rössler oscillators) have been considered; the corresponding precise CNN templates are derived to obtain precise solutions of corresponding equations. An implantation of the concept developed is possible even on embedded digital platforms (e.g. FPGA, DSP, GPU, etc.); this opens a broad range of applications. On-going works (as outlook) are using NAOP for deriving precise templates for a selected set of practically interesting PDE models such as Navier Stokes, Schrödinger, Maxwell, etc.**

Keywords: Stiff ODEs and PDEs, CNN-based differential equation solving, high-precision computing, ultra-fast computing, NAOP scheme for CNN templates' calculation.

I. INTRODUCTION

The last decades have witnessed a tremendous attention on solving nonlinear and stiff models (ODEs and/or PDEs) with the CNN paradigm [1]. The interest devoted to solving stiff models can be explained by their multiple potential applications especially in the so-called Computational Engineering context. Indeed, nonlinear models have been intensively used to understand, predict and describe the dynamical behavior of various engineering or natural systems. In the field of transportation and logistics, for example, traffic models do take the form of ODEs and/or PDEs [2]. Still, in the field of transportation, various image processing tasks which are of high importance for visual sensors in Advance Driver Assistant Systems (e.g. contrast enhancement, segmentation, edge detection, etc…) can be expressed through solving corresponding stiff ODEs and/or PDEs [3].

Diverse contributions have been made to develop analytical, numerical and even hardware-based approaches to solve stiff ODEs and/or PDEs [1]-[20]. Amongst these contributions some have retained our attention namely *"the solutions of PDEs and ODEs using the CNN-paradigm"*. In fact, the flexibility of the CNN paradigm and its huge potential to enable a renaissance of the old "analog computing" through an emulation on digital platforms (e.g. FPGA or GPU, etc.) to perform ultra-fast and accurate computing of nonlinear models are some of its strongest points. Nevertheless, the relevant state-of-the-art does not provide significant information related to a straight-forward method to calculate the CNN templates needed for solving stiff ODEs and/or PDEs with the CNN paradigm. Despite some intensive works developed in this direction it is still unclear how to solve PDEs and/or ODEs with good accuracy or high precision. Only approximate solutions exist, for example the use of CNN processors in an approximation of numerical solutions of PDEs involving the finite difference method [7], [10]-[14]. This late approach does not provide accurate results due to the Taylor series' expansion which does consider only up to the first order (i.e. linear expansion). A further interesting published approach to solve PDEs is the group of learning schemes involved in an approximated solution of PDEs through CNN processors

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[15]-[20]. This late approach does require some initial solutions along with some critical parameter settings of the equations under investigation in order to enable the training process. This is a clearly significant drawback as it is not always possible to provide this data/information whenever dealing with stiff ODEs and/or PDEs.

Our aim in this paper is therefore to contribute to the enrichment of the relevant state-of-the-art by proposing/developing a systematic methodology (based on the CNN paradigm) which should help to clear some of the problems actually unsolved by the classical above described approaches. The key challenge thereby is developing a CNN-based computing concept for performing both ultra-fast and high-precision computing of stiff differential equations. The proposed method is based on a nonlinear adaptive optimization scheme to which we give the acronym "NAOP". For proof of concept, the novel approach developed in this paper is applied to derive solutions of selected classical and well-known examples of stiff ODEs. In the following, the flexibility of the approach developed is extensively discussed and we then do show/explain an easy extension of this approach to similarly efficiently solving stiff PDEs.

The rest of the paper is organized as follows. Section 2 presents an in-depth description of the novel concept. The quintessence of NAOP is explained and we thereby describe the scheme for deriving appropriate CNN templates values for any given nonlinear ODE. Section 3 does then focus on the proof of concept through a selected nonlinear differential equation that is solved using the new concept developed in this paper: the van der Pol equation. For this, corresponding 'precise' templates are calculated through NAOP. In section 4 the possible extension of the novel scheme involving NAOP for solving PDEs is discussed. And finally, a series of concluding remarks are presented in Section 5 along with the presentation of some interesting open research questions (outlook) that are under investigation in some of our on-going works.

II. THE CONCEPT OF "NAOP" FOR CNN TEMPLATE CALCULATION FOR SOLVING STIFF ODES

This section describes the approach based on the Nonlinear Adaptive Optimization (NAOP) for solving ODEs. The overall flow diagram of this approach is schematically displayed by the synoptic representation in Fig. 1.

The NAOP is performed by a complex 'computing' "module/entity/procedure" which does work on two inputs. The first input contains wave-solutions of the models describing the dynamics of a CNN- network model built from state-control templates (see (1)):

$$\frac{dx_i}{dt} = -x_i + \sum_{j=1}^{M}\left[\hat{A}_{ij}x_j + A_{ij}y_j + B_{ij}u_j\right] + I_i \qquad (1)$$

The second input contains wave-solutions of the model or better the liner/nonlinear differential equation, under investigation which could be re-written in the following simplified form as a set/couple of second order ODEs (see (2)):

$$\frac{d^2y_i}{dt^2} = F(y_i, y_i^n, \dot{y}_i^m, z_i, z_i^n, \dot{z}_i^m, t) \qquad (2a)$$

$$\frac{d^2z_j}{dt^2} = F(z_j, z_j^n, \dot{z}_j^m, y_j, y_j^n, \dot{y}_j^m, t) \qquad (2b)$$

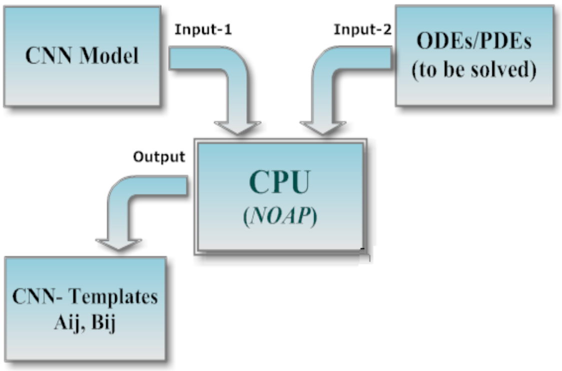

Figure.1. Synoptic representation of the key steps involved in the NAOP approach used for a precise template calculations for solving both linear and nonlinear differential equations.

The output of the NAOP system will generate, after extensive iterative computations or 'training' steps, appropriate CNN-templates to solve the corresponding ODEs (see (2)) when the convergence of the training process is achieved.

The global process to derive the CNN-templates (i.e. NAOP) can be summarized as follows. The learning/training process is based on a mapping between the two inputs of the NAOP procedure. A convergence to local minima is the key purpose governing this template calculation process, the so-called *NAOP*. To achieve this, various basins of attraction are investigated sequentially, and corresponding CNN templates are determined for those various initial conditions. If some local attractors diverge from a local minimum, new sets of initial conditions are automatically generated to annihilate the divergence leading to a possible convergence to a local minimum. A large number of randomly generated attractors (either regular or chaotic) are obtained through various numerical simulations whereby each attractor corresponds to a specific set of CNN-templates. An attempt to map these attractors to those generated by the model under investigation is performed in a sequential process leading to the convergence to a local minimum when the mapping is achieved successfully. However, it should be worth a mentioning that during the training process our various numerical simulations have revealed that it is very tough/difficult to find the optimal solution (i.e. the local minimum). This difficulty can be explained by the well-known inherent local minimum problem of the Hopfield neural network [8]-[9]. To overcome this problem, various basins of attractions are therefore generated within the *NAOP* process and this generation is conducted in a sequential way until the internal dynamics of the global network of coupled oscillators converges to stable states. This convergence must be achieved in both the ‘CNN-templates’ and the ‘attractors’ which are all considered to be dynamic variables during the learning/training process. It is further worth a mentioning that the quintessence of the concept *NAOP* is in the core an adaptive training process that is very comparable to the concept developed for the training of Hopfield neural networks towards an efficient tracking of local minima. Nevertheless, NOAP has been demonstrated capable of mapping all known nonlinearity of ODEs unto appropriate templates of a first-order CNN processor matrix.

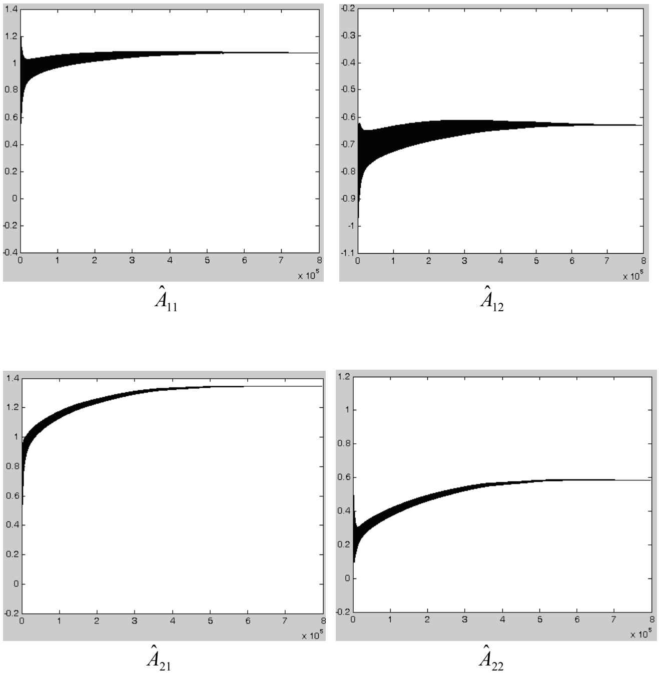

Figure. 2a: Convergence of state-control CNN templates as achieved by the NOAP process for the following values of the system parameters: Є =0.25 and ω=1.

III. APPLICATIONS TO SOLVING STIFF ODES

We restrict our analysis to the case of the van der Pol oscillator which is a good prototype of a well-known self-sustained oscillator having the interesting characteristic of being able to generate sinusoidal-, quasi-periodic-, and relaxation- oscillations (see (3))

$$\frac{d^2x}{dt^2} - \varepsilon\left(1 - x^2\right)\frac{dx}{dt} + \omega x = 0 \qquad (3)$$

Two possible states can be generated by (3). The first is the sinusoidal or almost sinusoidal state (Є <<1). The second one is the quasi-periodic state which could lead to relaxation oscillations for large values of Є (Є >>1). We now want to solve (3) using the CNN-paradigm. We envisage the case where Є=0.25 and ω=1. For these parameter values the *NAOP* concept has been exploited to calculate the corresponding templates after convergence of the training process. This convergence is clearly illustrated by the plots presented in Figs. (2a) and (2b) showing the temporal evolution of both the state-control templates $\hat{A}_{ij}$ (see Fig. (2a)) and the feedback templates A_{ij} (see Fig. (2b)). As it appears in these figures, the convergence is achieved after a long transient phase displayed by the global training network. It is worth a mentioning that the convergence of the process is achieved for suitable basins of attractions. From Fig. (2), one can easily read the following corresponding CNN templates that are then used to solve the van der Pol equation:

$$\hat{A}_{11} = 1.0770\ ,\ \hat{A}_{12} = -0.6300\ ,\ \hat{A}_{21} = 1.3450\ ,\ \hat{A}_{22} = 0.5850\ ,$$
$$A_{11} = 0.4473\ , A_{12} = -0.2586\ ,\ A_{21} = 0.4846\ ,\ A_{22} = 0.1310\ .$$

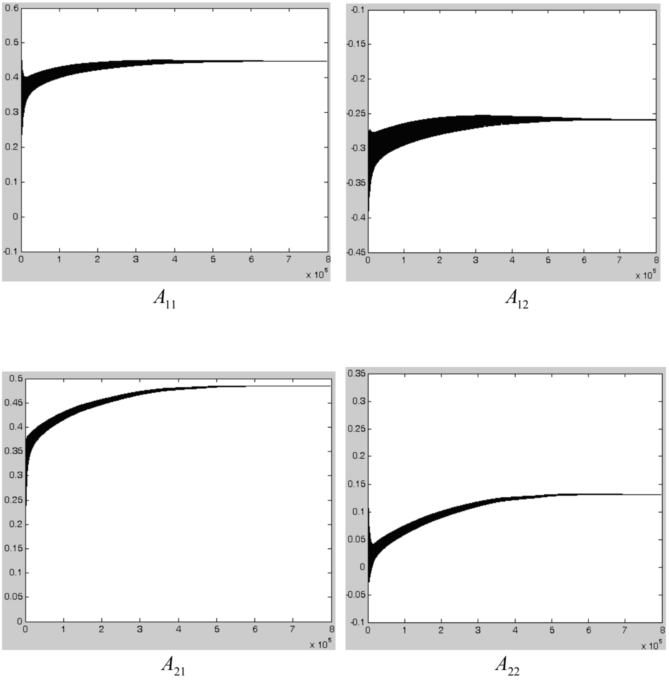

Figure. 2b: Convergence of Feedback- templates achieved by the NOAP process for the following values of the system parameters: Є =0.25 and ω=1.

This set of template values has been used/inserted in Fig. 3 to obtain the solution of (3) through the CNN paradigm. Indeed Fig. 3 is a general representation in SIMULINK of a CNN processor platform to solve second-order nonlinear ordinary differential equations. The key contribution of our approach, which is a breakthrough, is that we are now capable of transforming/ mapping any type of nonlinearity displayed by nonlinear coupled and uncoupled ODEs into the type of nonlinearity displayed by the elementary first-order CNN- cell model. As proof of concept of the approach developed in this paper, we have used the CNN templates derived by this scheme to obtain the exact solutions of (3). The graphical representation of the CNN-processors for second order ODEs presented in Fig.3 has been used for rapid prototyping purposes (a hardware implementation in either DSP or FPGA or GPU platforms is then straight-forward). A direct numerical simulation of the same equation, i.e. (3) has also been performed using MATLAB and a comparison between these two results is shown in Figs. 4. As it clearly appears in Fig. (4a), the result (i.e. the solution of (3)) by the approach based on the CNN-paradigm developed in this paper and the result (i.e. Fig. (4b)) of the same equation through a direct numerical solution through MATLAB of (3) are in a very good agreement (i.e. same value of the amplitude of oscillations and same frequency of the oscillations).

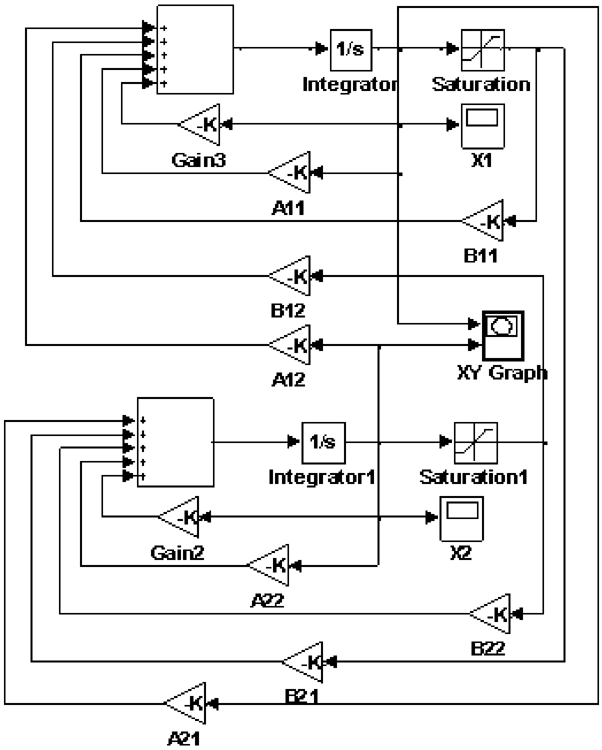

Figure. 3. SIMULINK graphical representation of the CNN- computing platform to solve (3).

The method proposed in this paper is challenging as it shows/demonstrates a systematic and straightforward way to solve nonlinear ordinary differential equations by the CNN-paradigm. The key challenge has been the possibility and then the appropriate way/algorithmic of/for mapping any type of nonlinearity unto the nonlinearity displayed by the elementary CNN- cell. Therefore, the approach developed in this work is very flexible as it can be applied to solve different types of nonlinear and stiff ODEs. The template calculation scheme based on *NAOP* has also been successfully applied for solving Rayleigh, Lorenz and Rössler equations and corresponding CNN- templates have been successfully derived (due to space constraints we cannot present all these results in this paper). One interesting issue under investigation is the establishment/development of a library of CNN template-sets to solve the most common nonlinear and stiff ODEs including the ones already cited above.

The next section is addressing the generalization/extension of the approach developed in this paper to solving nonlinear and stiff PDEs. In fact, it will be shown that a discretization process could help to transform PDEs into sets of coupled or uncoupled nonlinear ODEs in order to make them solvable by the CNN-paradigm while thereby applying the scheme developed in this paper.

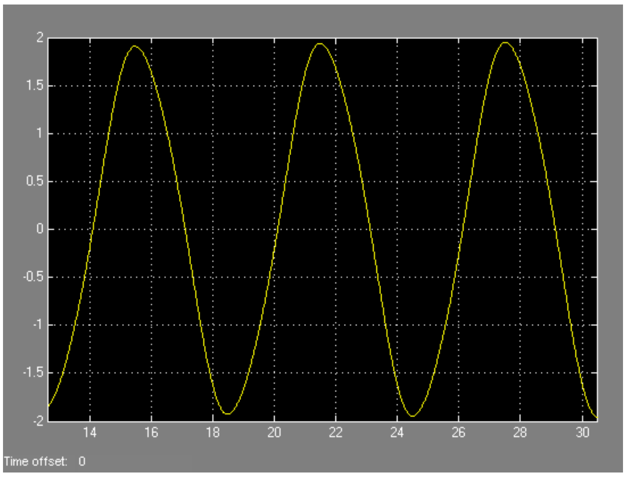

Figure 4a. Wave-form solution of (3) obtained by our new approach based on the CNN- paradigm for € =0.25 and ω=1.

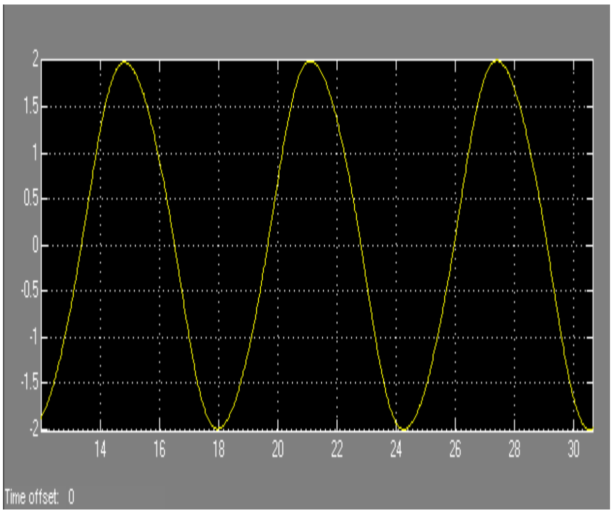

Figure 4b. Wave-form solution of (3) obtained through direct numerical simulation of (3) in MATLAB for € =0.25 and ω=1.

IV. EXTENSION OF THE NAOP SCHEME TO SOLVING STIFF PARTIAL DIFFERENTIAL EQUATIONS

This section explains the possibility of extending/applying the approach developed in this paper to solving PDEs. Unlike the traditional approach of solving stiff PDEs through CNN which takes into consideration only the linear terms of the Taylor's series expansion, we include the higher order derivative terms in the Taylor's series expansion of any given PDE in order to improve the accuracy of the obtained solutions. We consider, for illustration, the Burger's equation (4) which is a well-known prototype of partial differential equations and which is having multiple potential applications in the field of transportation.

$$\frac{\partial u}{\partial t} = \frac{1}{R}\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x} \qquad (4)$$

In order to solve (4) by the CNN-paradigm, applying an expansion (at the first order) based on the Taylor's series does lead to the following equivalent form of (4):

$$\frac{du_i}{dt} = \frac{1}{R}\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{u_i\left[u_{i+1} - u_{i-1}\right]}{2h} \qquad (5)$$

One can see that (5) is a well-known prototype of a set of first-order coupled nonlinear ODEs. As it appears in (5), the discretization performed has resulted into a set of coupled ODEs with quadratic nonlinear terms (i.e of types similar to Lorenz or Rössler). This type of nonlinearity is solvable by our approach *(NAOP)* developed in the preceding paragraph as we could already solve more complex types of nonlinearity (e.g. the nonlinearity in the van der Pol equation). As discussed in Section-I, taking the truncated Taylor's series (only the linear terms) has been done reluctantly in the many published works, since there has been no way so far, according to the literature, to deal with the increased complexity and the nonlinearity that appear otherwise. It is obvious that the results produced in the case of a linear approximation are de facto less precise. While considering the higher-order (in this case second-order) derivative terms in order to increase precision, the Taylor's series expansion could be applied to (4) and this could lead to results presented in (6):

$$\frac{du_i}{dt} = \frac{1}{R}\left[\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{u_{i+1} - 3u_i + 3u_{i-1} - u_{i-2}}{2h^2} - ....\right]$$
$$-u_i\left[\frac{u_{i+1} - u_i}{h} - \frac{u_{i+1} - 2u_i + u_{i-1}}{2h} - ......\right] \qquad (6)$$

Therefore, while considering (6), it becomes obvious that the *NAOP* developed in this paper is a best candidate for a straightforward derivation of the appropriate CNN-templates to solve (6).

NAOP is also applicable for solving PDEs. The PDE must be first transformed in a set of coupled nonlinear ODEs. In this process, even nonlinear terms can be kept. Then NAOP will be used to determine appropriate templates for solving those complex sets of generally coupled nonlinear ODEs.

V. CONCLUDING REMARKS

We have proposed and validated a theoretical/concept based on the CNN paradigm for ultra-fast, potentially low-cost and high-precision computing of stiff PDEs and ODEs. Since we can solve these through CNN independently of the actual nonlinearity, we have reached a clear breakthrough that has the potential to enable a really 'real-time' Computational Engineering.

The main benefit of solving PDEs and ODEs using CNN is the offered flexibility through NAOP to extract the CNN parameters through which CNN can solve any type of PDE or ODE. Another strong point of the CNN-paradigm is the then resulting ultra-fast processing depending on the CNN implementation: DSP, FPGA, GPU, or CNN-Chip. One key objective of this work has been to advance the relevant state-of-the-art by proposing a novel framework to solve stiff PDE's and ODE's with high- precision. To achieve this goal, we have proposed and demonstrated that the Nonlinear Adaptive Optimization (NAOP) technique is a best and efficient scheme to cope with solutions of any ODE or PDE. The NAOP is a learning/training method for mapping the wave solutions of the models describing the dynamics of a CNN-network to that of a given model (ODE). Taking just these two inputs, the learning process leads to the convergence to a local minimum where the complete mapping of the two models is achieved and CNN-templates are produced.

Using the same technique, we proposed a high- precision computing of stiff PDEs while accounting even nonlinear terms (i.e. high order-terms) in the Taylor's series expansion used while transforming the PDE unto a set of coupled nonlinear ODEs. In order to overcome the problem related to the speed of computation, an implementation either on FPGA or DSP or GPU of the concept developed in this work is possible and straight-forward.

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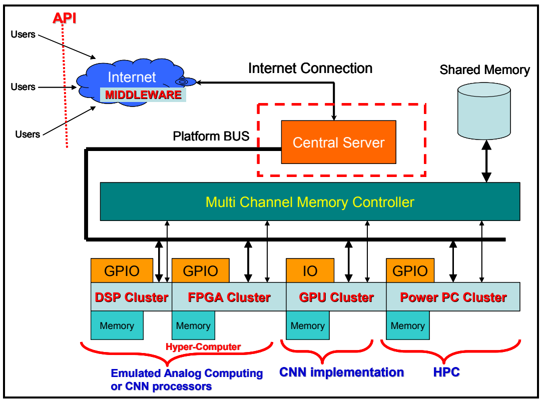

Figure 5. Global architecture of the computing platform planned to enable a real-time computational Engineering. Diverse users may access the CNN processor platforms in a remote way through the Internet

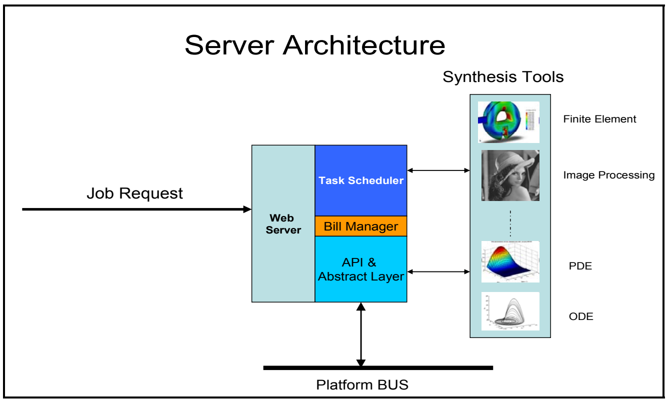

Figure 6. Core idea of the server architecture intended for the CNN based super-computing platform to enable real-time Computational Engineering. It is a detailed description of the central sever given in Fig. 5.