Dynamics of a Quasiperiodically Forced Rayleigh Oscillator

This paper studies the dynamics of a self-excited oscillator with two external periodic forces. Both the nonresonant and resonant states of the oscillator are considered. The hysteresis boundaries are derived in terms of the system parameters. The stability conditions of periodic oscillations are derived. Routes to chaos are investigated both from direct numerical simulation and from analog simulation of the model describing the forced oscillator. One of the most important contributions of this work is to provide a set of reliable analytical expressions (formulas) describing the system’s behavior. These are of great importance to design engineers. The reliability of the analytical formulas is demonstrated by a very good agreement with the results obtained by both the numeric and experimental analyses. [DOI: 10.1115/1.2232684]

Keywords: oscillatory states, hysteresis boundaries, stability criteria, nonlinear oscillator, chaos, bifurcations, analog simulation

1 Introduction

In recent years, a twofold interest has attracted theoretical, numerical, and experimental investigations to understand the behavior of nonlinear oscillators. The theoretical (fundamental) investigation reveals their rich and complex behavior, and the experimental (self-excited oscillators) describes the evolution of many biological, chemical, physical, mechanical, and industrial systems [1–3]. Recently, the chaotic behavior of these oscillators is exploited in the field of communication for coding information [4]. The periodically forced Rayleigh equation has been extensively studied in the investigation of the response of a self-excited oscillator when it is driven by a periodic force [2]. It is the first model of irregular oscillations (transient chaos or “ghost solutions”) of differential equations [5,6]. To the best of our knowledge, little has been done in a system consisting of a self-excited Rayleigh oscillator with two external periodic forces. The choice of an external force with two periodic components gives the possibility of different types of excitations (sinusoidal, quasiperiodic, time domain amplitude modulated, relaxative, and so on). This paper aims at the following: (a) considers the dynamics of the system; (b) contributes to the general understanding of the behavior of the system and points out some of its unknown behavior; and (c) experimental investigation of the dynamics of the system.

The dynamics of the forced Rayleigh oscillator is described by the following nonlinear differential equation:

$$\ddot{x} - e_1(1 - \dot{x}^2)\dot{x} + \omega^2x = f(t)$$  \hspace{1cm} (1a)

where $e_1$ and $\omega$ are positive parameters, being, respectively, the damping coefficient and the natural angular frequency.

Equation (1a) when $f(t) = 0$ exhibits a sinusoidal behavior for small $e_1$ and leads to a relaxation oscillation for large $e_1$. The latter is suited for the control in systems with input stimulus (that produces a response of fixed amplitude) but adaptable frequency or repetition rate. This is similar to a beating heart when each contraction of the ventricle is stimulated by a nerve impulse generated on contraction of the atricle [3]. A self-excited Rayleigh oscillator can also be used in communication in its autonomous mode for small $e_1$, as a sinusoidal oscillator amplitude control, the
natural frequency $\omega$ being the control parameter. A sinusoidal function generator is an important circuit that is used in measurements, instrumentation, telecommunications, and electronics, to name a few.

In the presence of an external excitation, interesting phenomena (phase locking, collapse of torus, and folded-torus) are observed [6]. In this paper we concentrate on the analysis of Eq. (1c) when

$$f(t) = k_1 \cos(\omega t + \theta_1) + k_2 \cos(\omega_2 t + \theta_2)$$  \hspace{1cm} (1b)

Reference [7] considers the analytical approach to torus bifurcations and [8], the analytical and numerical study of the Duffing oscillator subjected to two external periodic forces.

The paper is structured as follows. Section 2 gives an analytical treatment of Eqs. (1). Approximate solutions of Eqs. (1) in the nonresonant case are obtained with the multiple time-scales method [1]. Considering the resonant case, we analyze the effects of the amplitudes $k_1$ and $k_2$ and those of the detuning parameter on the oscillator’s behavior. Using the perturbation method [2], the stability conditions of periodic oscillations in a forced Rayleigh oscillator are derived. Section 3 evaluates the direct numerical integration of Eqs. (1) in both the nonresonant and resonant cases, and the oscillatory states are analyzed. It ends with an analysis on the effects of the amplitude $k_1$ and the frequency $F_1$ on the oscillator’s behavior. The degree of chaos is characterized by calculating the largest one-dimensional (1D) numerical Lyapunov exponent and determining the bifurcation diagram. Section 4 investigates using analog simulator, the dynamics of the system described by Eqs. (1). Here we present the electrical circuit of our simulator. The definition of the parameters of Eqs. (1) as functions of the circuit components are presented as well. The section ends by comparing the results from our simulator to those of the direct numerical integration of Eqs. (1). Section 5 deals with conclusions and proposals for further work.

2 Analytical Treatment

We seek approximate solutions of Eqs. (1) by the well-known multiple time-scales method [1].

2.1 Nonresonant Case. Here, $\omega_1$ and $\omega_2$ are incommensurable.

Assuming a small nonlinearity, from the zero-order perturbation theory, we obtain from Eqs. (1):

$$x(t, \epsilon) = \left( \frac{8 \eta_0^3}{3 \omega_1^3} \right)^{1/2} \left[ 1 - \left( \frac{8 \eta_0 k_2 \omega_2^2}{3 \omega_1^3} \right)^{1/2} \right]^{1/2} \cos(\omega t + \theta)$$

$$+ \sum_{i=1}^{2} \frac{k_i}{\omega_2 - \omega_1} \cos(\omega_i t + \theta_i)$$  \hspace{1cm} (2a)

where $\eta_0$ stands for the initial amplitude, $\epsilon$ is a small dimensionless parameter, and

$$\eta = 1 - \frac{3}{4} \frac{k_2}{\omega^2 - \omega_1^2}$$  \hspace{1cm} (2b)

Hence, the nature of solutions depends on a critical relationship between the parameters of Eqs. (1). When $\eta > 0$, the steady-state motion is a superposition of three terms of the frequencies $\omega_1$, $\omega_2$, and $\omega_3$ with amplitudes $(3 \omega_1^2)^{1/2} (8 \eta_0^2)^{1/2}$, $(\omega_2^2 - \omega_1^2)^{1/2} k_1$, and $(\omega_2^2 - \omega_1^2)^{1/2} k_2$, respectively. Since $\omega_1$ and $\omega_2$ are incommensurable, the motion is aperiodic. When $\eta < 0$, the steady-state motion consists of the forced solutions only. Under this condition, the free oscillations tend to zero with increasing time, leading to the quenching of self-sustained oscillations.

The result when $\eta < 0$ is not interesting, since it corresponds to the case when the self-sustained oscillations are quenched and consequently have no effect on the wave generated by the oscillator. This justifies the interest devoted to the resonant case as analyzed below.

2.2 Resonant Case and Hysteresis Boundaries. We have analyzed subharmonic and superharmonic resonances and found that only the primary resonance is interesting. We then restrict our analysis to the interesting case when $\omega = \omega + \sigma \epsilon$, where $\epsilon_1$ and $\epsilon_2$ are assumed small, $\omega_1$ and $\omega_2$ are incommensurable and $\sigma = O(1)$.

It is found using the multiple time-scales method that the zero-order perturbation theory yields the general solution of Eqs. (1)

$$x(T) = \frac{1}{2} a(T) \exp\left[ j(\omega T + \beta(T)) \right] + \frac{k_2}{2(\omega^2 - \omega_2^2)} \exp\left[ j(\omega_2 T + \beta_2) \right]$$

$$+ c.c.$$  \hspace{1cm} (3)

The amplitude $a(T)$ is the solution of the following differential equations:

$$\frac{da}{dT} = -\frac{3}{8} \epsilon \omega_1 \omega_2^2 \left( \frac{\eta_0^2}{3 \omega_1^3} \right)^{1/2} \sin \theta$$  \hspace{1cm} (4a)

$$\frac{d \theta}{dT} = -\frac{k_2}{2 \omega_2} \cos \theta$$  \hspace{1cm} (4b)

$$\eta = \frac{1}{2} \frac{3 \omega_1^2 k_2^2}{4(\omega_1^2 - \omega_2^2)^2}$$  \hspace{1cm} (4c)

Thus, at resonance $\omega_1 = \omega + \sigma \epsilon$, the steady-state motion is described by Eq. (3) with the amplitude $a(T_1)$ obtained from the following nonlinear algebraic equations:

$$\left( \frac{\omega_1^3}{64} \right) a^3 - \left( \frac{\eta_0}{12} \right) a^4 + \left( \frac{\eta_0^2}{9 \epsilon_1^2 \omega_1^6} \right) a^2 - \frac{k_2^2}{3 \epsilon_1^2 \omega_1^6} = 0$$  \hspace{1cm} (5)

From Eq. (5), two types of solutions are expected: the first is within the hysteresis domain and the second is out of the domain. Thus, the general solutions within the hysteresis domain yield:

$$a = \frac{4 \sqrt{\eta} \sqrt{1 - \frac{3 \sigma}{\epsilon_1}} \cos \left( \theta + 2 \pi \ell \right)}{3}$$  \hspace{1cm} (6a)

while that out of the domain yields

$$a = \sqrt{\left( \frac{q}{2} + \sqrt{\Delta} \right)^3 - \left( \frac{q}{2} + \sqrt{\Delta} \right) \frac{q}{2} + \frac{q^3}{16 \eta_0 \omega_1^6} + \frac{16 \eta_0 \omega_1^6}{12 \eta_0 \omega_1^6} \theta}$$  \hspace{1cm} (6b)

Here,

$$\theta = \arccos \left\{ \left[ -1 - \frac{3 \sigma}{\epsilon_1} \right]^{1/2} + \frac{81 k_2^2}{64 \eta_0^2 \epsilon_1^4} \left[ 1 - \frac{3 \sigma}{\epsilon_1} \right]^{-3/2} \right\}$$  \hspace{1cm} (6c)

$$q = \frac{8}{9 \omega_1^2} \left[ 1 + \frac{3 \sigma}{\epsilon_1} - \frac{81 k_2^2}{64 \eta_0^2 \epsilon_1^4} \left[ 1 - \frac{3 \sigma}{\epsilon_1} \right]^{-3/2} \right]$$  \hspace{1cm} (6d)

$$\Delta = \frac{8 \eta_0^2}{9 \omega_1^2} \left[ -1 + \frac{3 \sigma}{\epsilon_1} \right]^{1/2} + \frac{81 k_2^2}{64 \eta_0^2 \epsilon_1^4} \left[ 1 - \frac{3 \sigma}{\epsilon_1} \right]^{-3/2}$$  \hspace{1cm} (6e)

and $\ell$ is an integer.

The graphical representation in Fig. 1 summarizes the effects of $\sigma$, $k_1$, and $k_2$ on the occurrence of the hysteresis phenomenon. Each region $\Sigma_1$, $\Sigma_2$, and $\Sigma_3$ has two parts, symmetric with respect to the $k_1$-axis, $\sigma k_1$, and $k_2$ planes show various regions (black regions) in which hysteresis phenomena occur and depend on some relationships between the parameters (in Eqs. (1)) deductible from Fig. 1. It is observed that the dimensions of these regions are dependent on values of the model’s parameters and are very sensitive to small changes in $k_2$. The white regions denote nonhysteresis phenomena. It is shown that there exist some critical values of $k_2$ (for example, say $k_2 = \epsilon_1^2$) for which the nonhysteresis regions
Fig. 1 Graphical representation of the hysteresis domains (black regions) and nonhysteresis domains (white regions) in the \(\rho, k_1\) plane (\(\Sigma_1, \Sigma_1', \) and \(\Sigma_2\)) and \(k_1, k_2\) plane (\(\Sigma_3\)): \(\Sigma_1\) (\(k_2 = C_G, \) and \(l_1 = (8/9)e_1, 2\gamma \)), \(\Sigma_1'\) (\(k_2 = C_a, \) and \(l_1 = (8/9)e_1, 2\gamma \)), \(\Sigma_2\) (\(k_2 = C_G, l_2 = (8/9)e_1, 2\gamma \)), \(\Sigma_3\) (\(k_2 = C_a, l_2 = (8/9)e_1, 2\gamma \)), and \(l_2 = (8/9)e_1, 2\gamma \)) and \((\Sigma_3')\))

\[
\begin{align*}
\sigma_0 &= \eta e_1 \sqrt{\frac{2}{3}} \sqrt{1 + \frac{81k_1^2}{32\eta e_1^3} \cos \left(\frac{\theta_0}{3}\right) - 1} \\
\sigma_1 &= \eta e_1 \sqrt{\frac{2}{3}} \sqrt{1 + \frac{81k_1^2}{32\eta e_1^3} \cos \left(\frac{\theta_0 - 2\pi}{3}\right) - 1} \\
\theta_0 &= \arccos \left\{ \frac{1 - \frac{405k_1^2}{64\eta^2 e_1^3} - \frac{6561k_1^2}{8192\eta^2 e_1^3} \left[1 + \frac{81k_1^2}{32\eta^2 e_1^3}\right]^{-3/2}}{1 + \frac{81k_1^2}{32\eta^2 e_1^3}} \right\}
\end{align*}
\]

It is found from \(\Sigma_1\) that the hysteresis phenomenon disappears when \(k_1 = (8/9)k_1' \sqrt{4\eta}\). At perfect resonance (\(\sigma = 0\)), the windows for the occurrence of the hysteresis phenomenon can be expressed from \(\Sigma_3\) in terms of \(k_1\) and \(k_2\). For \(\sigma \neq 0\), it is very difficult to derive analytically (in terms of \(k_2\)) the hysteresis boundaries.

From Eq. (5), we determine the amplitudes \(a\) (for varying \(\sigma, k_1,\) and \(k_2\)) that assist in verifying the analytical hysteresis boundaries and investigating the effects of \(\sigma, k_1,\) and \(k_2\) on the steady-state motion. Figure 2 shows the frequency-response curves in the \(\sigma\) vs. plane for some selected values of \(k_1\) when \(\omega = 1.0000, e_1 = 0.1312, \omega_2 = 2.3.0000,\) and \(k_2 = 1.00000\). Figure 3 shows (in the \(k_1\) vs. plane) the amplitude-response curves for some selected values of \(\sigma\) when \(\omega = 1.0000, e_1 = 0.150, \omega_2 = 2.3.000,\) and \(k_2 = 0.0465\). Figure 4 shows (in the \(k_2\) vs. plane) the amplitude-response curves for some selected values of \(\sigma\) and \(k_1\) when \(\omega = 1.000, e_1 = 0.150,\) and \(\omega_2 = 2.3.000\).

From Fig. 2 is observed the hysteresis phenomenon for \(k_1 = 0.015, k_1 = 0.025,\) and \(k_1 = 0.030\). For \(k_1 < 0.025\), the curves shown in Fig. 2 yield two branches: the first (lower) branch is a resonance curve and the second (upper) is a closed curve approximated to an ellipse with center \(\Omega_1 = (0, \alpha_1)\).

Fig. 2 Effects of \(\sigma\) and \(k_1\) on the resonant frequency—response curves (\(\omega = 1, \omega_2 = 2.3, e_1 = 0.15,\) and \(k_2 = 0.065\)) Analytical results (solid lines) and numerical results (stars) for \(k_1 = 0.15; k_1 = 0.025; k_1 = 0.030;\) and \(k_1 = 0.055\). Stability boundary (squares) and unstable region (shaded)

\[
\alpha_1 = \frac{2\sqrt{\eta}}{3} \left\{ \sqrt{1 + \cos \left(\frac{\phi_1}{3}\right)} + \sqrt{1 + \cos \left(\frac{\phi_1 - 2\pi}{3}\right)} \right\}
\]

and

\[
\phi_1 = \arccos \left\{ -1 + \frac{81k_1^2}{64\eta e_1^3} \right\}
\]

Equations (8) show the downshift and shift of each ellipse's center with increasing \(k_1\). With further increases in \(k_1\), the ellipses expand and open at \(k_1 = (8e_1/9)\sqrt{4\eta}\). Finally, branches intersect at points to give common branches in which the hysteresis phenomenon is still observed. When \(k_1 = (8e_1/9)\sqrt{4\eta}\), the hysteresis phenomenon disappears and the response curves are single-valued for all

Fig. 3 Effects of \(k_1\) and \(\sigma\) on the resonant amplitude—response curves (\(\omega = 1, \omega_2 = 2.3, e_1 = 0.15,\) and \(k_2 = 0.0465\)): Analytical results (solid lines) and numerical results (stars) for \(\sigma = 0.000; \sigma = 0.002; \sigma = 0.003;\) and \(\sigma = 0.005\) Stability boundary (squares) and unstable region (shaded)
\[
\left( \frac{2}{3\sigma} \right)^{-2} + \left( \frac{\sqrt{2}a^2 - \omega_0^2}{\Omega^2} \right)^{2} - 1 = k^2
\]

(10)

(see the lower line with squares for \(a=0\)). As \(k_1\) increases further, the ellipses are subjected to a "compression phenomenon" and finally reduce to a single-point \(\Omega_k(0, 2/3\sigma)\) when \(k_1 = (4/9)k_3\). Under this condition, the response-curves are single-valued for all \(k_2\) (\(k_2 \neq 0\)).

We now analyze the effects of \(k_2\) on the steady-state motion when \(\sigma = 0.057191752, \sigma = 0.002189636, \sigma = 0.02872350, \sigma = 0.005385039, \) and \(\sigma = 0.003919000\). The curves for \(\sigma = 0\) are pairwise conjugates as indicated by the notations (i.e., the upper four curves \(C_1, C_2, C_3,\) and \(C_4\) are, respectively, pairwise conjugates with the four lower ones; \(C_1^*, C_2^*, C_3^*,\) and \(C_4^*\) (\(C_2^*\) is a single point as mentioned above)). When \(\sigma = 0.001391752\), the curves \(C_1\) and \(C_4\) intersect at \(P_1(0, 1.144)\) to form a common curve \(C_1^*\). For \(\sigma = 0.002189636\), there is a shift in \(P_1\) and also a deformation of the piecewise curve having that point. In addition, the curves \(C_2\) and \(C_3^*\) intersect at \(P_2(0, 1.128)\) to form a common curve \(C_3^*\). At the value \(\sigma = 0.02872350\), a shift in \(P_3\) and also a deformation of the piecewise curve having that point is observed. In addition, the curves \(C_3\) and \(C_4\) intersect at \(P_3(0, 1.108)\) to form a common curve \(C_4^*\). It would be of interest to note that when the parameter \(\sigma\) is chosen in the window \(0.003103200 < \sigma < 0.005385039\), the single point \(C_4^*\) becomes a closed curve (an ellipse). When \(\sigma\) increases up to the boundary value \(0.003585039\), the curves \(C_4\) and \(C_4^*\) intersect at \(P_4(0, 1.067)\) to form a common curve \(C_4^*\).

Here, there is a shift in \(P_3\) and also a deformation of the piecewise curve having that point. When \(\sigma\) increases (Fig. 4), the hysteresis domain shrinks and finally disappears at \(\sigma = 0.003919000\) and all the four curves \(C_1^*, C_2^*, C_3^*,\) and \(C_4^*\) become resonance curves.

The results in this section have shown the possibilities of obtaining one, two, or three real values of the amplitudes \(a\). This has raised the question of the stability of these solutions and therefore justifies the interest devoted to Sec. 2.3.

2.3 Stability of Periodic Oscillations. The stability of periodic oscillations is investigated in the \(\sigma, k_1, k_2,\) and \(\sigma_2\) planes. We use first the perturbation method \(x = x(t) + \xi\) to obtain the variational form of Eqs. (1). The dissipative coefficient of the variational equation is expressed in terms of the solution \(x\) defined in Eq. (3). Letting \(\xi = y(t)\exp(-\omega t)\), \(\alpha(t)\) and \(y(t)\) being real functions, it is found that periodic solutions \(y(t)\) are obtainable from the following equation:

\[
\dot{y} + \left[ \omega^2 - \frac{e_1^2}{4}(1 - 3\lambda^2)^2 - 3\epsilon_x \dot{y} \right] y = 0
\]

(11a)

where

\[
u(t) = \delta + \text{additional periodic terms}
\]

(11b)

and

\[
\delta = -\frac{e_1}{2} \left[ 2\eta - 3\omega^2 \lambda^2 \right]
\]

\[
\eta \text{ is obtained from Eq. (4c). Assuming that the characteristic exponent is associated with the first and second unstable regions [2],}
\]

we can establish the following Hill's equation (considering Eqs. (3) and (11)):

\[
\dot{y} + \left[ \theta_0 + 2 \sum_{m=1}^{2} \theta_{2m+1} \cos(2\omega_{2m+1}t - \epsilon_{2m}) \right] y = 0
\]

(12)

In its derivation, terms of \(e_1\) higher than second order are ignored since our analysis is restricted to the case of small nonlinearity. The quantities \(\theta_0, \theta_1, \theta_2, \theta_3,\) and \(\epsilon_{2m}\) are expressed in terms of the parameters of Eqs. (1). The oscillations are stable if the perturbation \(\xi(t)\) tends to zero with increasing time. Thus, the quan-
lity $\delta$ must be positive and greater than the characteristic exponent (assumed positive and calculated by the Whittaker's method) of the solution of Eq. (12) [2]. Hence, the stability conditions are given (in terms of the parameters of Eqs. (1)) by the inequalities

$$3\alpha_1^2\varepsilon^2 - 4\eta > 0 \quad (13a)$$

and

$$\varepsilon^2(\alpha_1^2 - \alpha_0^2) \left[ \frac{9\omega_0^4\alpha_0^4}{8} + \frac{9}{8} \frac{k_2\omega_0^4}{\omega_0^4 - \omega_0^4} \right] + \varepsilon^2(\alpha_1^2) \left[ 2\eta - \frac{3\alpha_1^4\varepsilon^2}{2} \right] + \omega_0^2(\alpha_1^2) > 0 \quad (13b)$$

Thus, at resonance ($\omega_1 = \omega + \varepsilon\sigma$) relations (13) show that the unstable portions of the response curves are located in the shadow areas of Figs. 2–4.

### 3 Numerical Computation

The aim of the numerical study is to verify the analytical results established in Sec. 2, to find the sensitivity and some sets of parameters leading to chaotic behavior and also to define routes to chaos.

#### 3.1 Oscillatory States

We restrict our analysis to the verification of the analytical results obtained in the resonant case. We use the discrete Fourier transform (DFT) to determine (from Eqs. (1)) numerical values of the amplitudes $\alpha$ in the resonant state, and some numerical results are provided (see star lines in Figs. 2–4) for the same values of the system parameters in Sec. 2. Figure 2 shows the numerical frequency-response curves in terms of the detuning parameter $\sigma$ both for $k_1 = 0.015$ and $k_1 = 0.050$. From this figure, there is no agreement between the analytical and numerical results, although a divergence is observed for some points. This divergence can be explained by the occurrence of the harmonic frequency entrainment phenomenon in the quasiperiodically forced Rayleigh oscillator. Figure 3 shows some numerical points of the amplitude-response curves in terms of $k_1$ when $\sigma = 0.003$ and $\sigma = 0.005$. It clearly shows (in the domain of stable oscillations) a good agreement between the analytical and numerical results. We found a divergence between the results for both methods in the domain of unstable oscillations when $\sigma = 0.003$. In Fig. 4 some numerical points of the amplitude-response curves are shown in terms of $k_2$ when $k_1 = 0.005$ and $k_1 = 0.008$. Here a very good agreement (in the stability regions) between numerical and analytical results and a divergence (in the instability regions) between the two methods, are clearly shown. It should be noted that the rate of divergence decreases with increasing $\sigma$. Hence, this divergence can also be explained by the occurrence of the harmonic frequency entrainment phenomenon in the forced Rayleigh oscillator.

#### 3.2 Chaotic Behavior

Here, the types of motion are identified using two indicators. The first indicator is the bifurcation diagram, the second being the largest 1D numerical Lyapunov exponent denoted by

$$\lambda_{\text{max}} = \lim_{t \to \infty} \frac{\ln|d(t)|}{t} \quad (d(t) = \sqrt{\varepsilon^2 + \left( \frac{d\varepsilon}{dt} \right)^2}) \quad (14)$$

and computed from the variational equation obtained by perturbing the solution of Eqs. (1) as follows: $x \rightarrow x + \varepsilon d(t)$ is the distance between neighboring trajectories. Asymptotically, $d(t) = e^{\lambda_{\text{max}}t}$. Thus, if $\lambda_{\text{max}} > 0$, neighboring trajectories diverge and the state is chaotic. If $\lambda_{\text{max}} < 0$, these trajectories converge and the state is non-chaotic. $\lambda_{\text{max}} = 0$ for the torus states [15]. Setting $\omega = 1$ and $k_2 = 0$, we analyze the effects of the amplitude $k_1$ and the frequency $F_1$ on the behavior of the Rayleigh oscillator. Therefore, a scanning process is performed to investigate the sensitivity

![Fig. 5](image1)

**Fig. 5** Bifurcation diagram showing the coordinate $x$ of the attractor in the Poincaré cross section versus $k_1$ ($\omega = 1.000$, $k_2 = 0.000$, $\gamma = -2.300$, and $F_1 = 0.040$)

of the oscillator to tiny changes in $k_1$ and $F_1$. The investigation is carried out in the following windows: $1.4 < k_1 < 2.4$; $2.4 < k_1 < 2.5$; $2.5 < k_1 < 3.0$; $3.0 < k_1 < 3.5$; and $0.0275 < F_1 < 0.0425$.

Considering the effects of $k_1$, it appears that the oscillator described by Eqs. (1) leads to complex dynamical behavior, such as torus, multiperiodic, quasiperiodic, and chaotic states. We observed various routes to chaos (such as sudden transition, period-doubling, torus breakdown, or quasiperiodic routes) with several kinds of periodic and multiperiodic windows. Figure 5 provides some sample results, showing the bifurcation diagram ($k_1$, $x$) (Fig. 5(a)) associated to its corresponding graph of largest 1D numerical Lyapunov exponent (Fig. 5(b)) when $\varepsilon_1 = 2.300$ and $F_1 = 0.040$. A torus breakdown route to chaos is shown (Fig. 5). Figure 6 shows some numerical phase portraits chosen in the window $1.4 < k_1 < 2.4$; Fig. 6(a) shows torus ($k_1 = 1.850$) and Fig. 6(b) chaotic ($k_1 = 2.398$) phase portraits. Period-doubling, period-doubling, and period-5 sudden transition routes to chaos are also found in the range $3.0 < k_1 < 3.5$. Here, windows of regular motions and small windows of chaotic motions appear and disappear alternately. Samples phase portraits are obtained confirming the sequence of bifurcation (for example, period-2 ($k_1 = 3.0100$), chaotic ($k_1 = 3.1635$) and period-1 ($k_1 = 3.1800$) phase portraits).

A similar scanning process was performed to see how the frequency $F_1$ of the first component of the external excitation affects

![Fig. 6](image2)

**Fig. 6** Numerical phase portraits of the oscillator with the parameters of Fig. 5 for $k_1 = 1.850$ (torus) and $k_1 = 2.398$ (chaos)
the behavior of the oscillator. We thus set $\varepsilon_1 = 1.3900$ and $k_1 = 3.3280$. Figure 7 shows the bifurcation diagram of the attractor $x$ (Fig. 7(a)) with the corresponding graph of the largest 1D numerical Lyapunov exponent (Fig. 7(b)) when $F_1$ is monitored. It shows a period-adding (period-1 $\rightarrow$ period-2 $\rightarrow$ period-3 $\rightarrow$ chaos) route to chaos. Also shown is the extreme sensitivity of the oscillator to small changes in $F_1$. Windows of regular motions randomly alternate with windows of chaotic motions. Figure 7 also shows period-6 and period-8 sudden transition routes to chaos. These transitions are clearly observed when $F_1$ is scanned between 0.0280 and 0.0300. In Fig. 8 are shown some numerical phase portraits: Period-1 (Fig. 8(a): $F_1 = 0.0410$), period-2 (Fig. 8(b): $F_1 = 0.0385$), period-3 (Fig. 8(c): $F_1 = 0.0360$), and chaotic (Fig. 8(d): $F_1 = 0.0275$) phase portraits are observed.

Different routes to chaos observed in a periodically forced Rayleigh oscillator are commonly observed in nonlinear systems, such as forced systems, coupled autonomous systems, and coupled forced systems [8,9,11,13,15], to name a few. This serves to justify the richness of the bifurcations in a periodically forced Rayleigh oscillator.

4 Analog Computation

The analog computer implementation is a nice way to scan the parameter range in order to find the proper parameter values for a numerical simulation. Another advantage of such an implementation with respect to numerical simulation is that there is no need to wait for long transient times. This justifies the increasing interest devoted to this type of implementation for the analysis of nonlinear and chaotic physical systems [10-13].

Our aim in this section is to consider and implement an appropriate analog simulator for the investigation of the system described by Eqs. (1). We here restrict our analysis to the study of different bifurcations and onset of chaos.

4.1 Design of the Analog Simulator and Resulting Parameters of Eqs. (1). Figure 9 shows the electronic simulator used to simulate Eqs. (1). The sine generators $e_i$ ($i=1,2$) are the external excitations. The electronic multipliers $M_i$ and operational amplifiers $U_i$ are Analog Devices (see Ref. [13] for the type, the pin numbers and the offset voltages cancellation). Using an appropriate time scaling, the simulator outputs can be viewed directly on an oscilloscope. The phase portrait $(x, \dot{x})$ with $\alpha = 10^4 C_1 R_1 R_2 C_2$, $\beta = 10^8 C_1 R_1 R_2 C_2$ is obtained by feeding the output voltages of the operational amplifiers $U_i$ and $U_j$, respectively, to the $X$- and $Y$-inputs of the oscilloscope.

In terms of the circuit components, the parameters of Eqs. (1) are defined as follows:

$$\omega = 10^{-4}(R_1 R_2 C_2)^{-1/2}; \quad \varepsilon_1 = 10^{-4} R_3 (R_1 R_2 C_2)^{-1};$$

$$k_1 = 10^{-8} \varepsilon_1 (R_1 R_2 C_2)^{-1};$$

$$k_2 = 10^{-8} \varepsilon_2 (R_1 R_2 C_2)^{-1}; \quad \varepsilon_2 = 10^{-8} (\omega_0 \tau + \theta);$$

The time unit is $10^{-4}$s and $R_0 = 10^8 R_1 R_2 C_2$ (see Ref. [13] for the complete derivation technique).

4.2 Bifurcation and Onset of Chaos. This section is devoted to the experimental findings. Various bifurcations and types of motion are examined when one component of the electronic circuit is monitored. The control components are the resistor $R_3$ (or
Table 1 Comparison of the bifurcation values for both numerical and experimental computations

<table>
<thead>
<tr>
<th>Control Parameters</th>
<th>Transitions (m → n)</th>
<th>Bifurcation Values</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>From Period m</td>
<td>To Period n</td>
</tr>
<tr>
<td>Torus</td>
<td>5</td>
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<tr>
<td>5</td>
<td>Chaos</td>
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<tr>
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<td>Chaos</td>
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<td>2</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>Chaos</td>
</tr>
</tbody>
</table>

The parameter $k₁$ and the frequency $F₁$ of the excitation.

In order to control each parameter of Eqs. (1) by varying only one resistor, we set the following values: $R₁=9990$ Ω; $R₂=1002$ Ω; $R₃=9970$ Ω; $R₄=9957$ Ω; $C₁=10.01$ nF; $C₂=10.01$ nF; $\alpha₁=(1/\sqrt{2})V_{ref}$. Thus, the coefficients $\alpha$, $\beta₁$, $k₁$, and $k₂$ will respectively be controlled by $R₄$, $R₃$, $R₇$, and $R₈$. It is important to mention that the analog voltages obtained from our simulator are directly equivalent to the dimensionless variable $x$ of Eqs. (1).

We now experimentally investigate the effects of both amplitude $k₁$ (when $F₁=0.0400$ Hz; $R₃=2300$ Ω; $R₄=9990$ Ω; $R₃=\infty$) and frequency $F₁$ (when $R₃=1390$ Ω; $R₄=9990$ Ω; $R₇=3000$ Ω; $R₈=\infty$) on the bifurcations exhibited by the Rayleigh oscillator by using the values of resistors corresponding exactly to the same values of parameters in Sec. 3.2. Some sample results are provided in Table 1. It is found that the oscillator exhibits complex bifurcation structures such as period-$n$ ($n=1, 2, 3, \ldots$), torus, multi-periodic, and chaotic bifurcations with several windows of chaotic motion alternating with windows of regular motion. The extreme sensitivity of the oscillator’s behavior to small changes in $k₁$ and $F₁$ is also found. Various bifurcation points are obtained and some samples are shown. A comparison of these bifurcation points with the results from the numerical analysis (Table 1) shows a good agreement, although a small divergence is observed and increases with increasing $k₁$. This can be explained by the precision on the values of the components of the electronic circuit.

Various experimental phase portraits are obtained and compared to those from the numerical computation, and a very good similarity is observed. Figure 10 shows two samples of experimental phase portraits obtained for the oscillator $x$ when the control component $R₇$ ($k₁$) is monitored. $P₁$ and $P₂$ are respectively the images obtained when $R₇=5400$ Ω and $R₇=4166$ Ω (Fig. 10). The system follows the following bifurcation when the resistor $R₇$ decreases ($k₁$ increases): torus bifurcation ($P₁$) → chaotic bifurcation ($P₂$).

The frequency $F₁$ of the excitation $e₁$ is now monitored in order to study its effect on the bifurcations of the attractor $x$. $F₁$ is

![Fig. 10 Experimental phase portraits of the oscillator with the parameters of Fig. 6: $P₁$: $R₇=5400$ Ω ($k₁=1.850$) (torus) and $P₂$: $R₇=4166$ Ω ($k₁=2.388$) (chaos). Voltage scales: X-input: 1 V/div. and Y-input: 1 V/div.](image-url)
Fig. 11 Experimental phase portraits of the oscillator with the parameters of Fig. 8: \( P_1: F_1 = 0.0410 \) Hz (period-1), \( P_2: F_1 = 0.0385 \) Hz (period-2), \( P_3: F_1 = 0.0360 \) Hz (period-3), and \( P_4: P_3 = 0.0275 \) Hz (chaos). Voltage scales: X-input: 1 V/div. and Y-input: 0.5 V/div.

monitored in the following window \( 0.0275 \text{ Hz} \leq F_1 \leq 0.0410 \text{ Hz} \). \( P_3, P_4, P_5, \) and \( P_6 \) are respectively the images obtained when \( F_1 = 0.0410 \text{ Hz}, \, F_1 = 0.0385 \text{ Hz}, \, F_1 = 0.0360 \text{ Hz}, \) and \( F_1 = 0.0275 \text{ Hz} \) (Fig. 11). The system follows the following bifurcations as \( F_1 \) decreases: period-1 bifurcation \( (P_3) \rightarrow \) period-2 bifurcation \( (P_4) \rightarrow \) period-3 bifurcation \( (P_5) \rightarrow \) chaotic bifurcation \( (P_6) \). This sequence of bifurcations shows period-adding transition route to chaos.

5 Conclusion

This work investigates the dynamics of the forced Rayleigh oscillator. Analytical, numerical, and experimental methods were used for the investigation. Analytical formulas describing the behavior of the forced oscillator were derived both in the nonresonant and resonant cases. It was found that the forced oscillator exhibits hysteresis, resonance, frequency entrainment, and quenching phenomena of self-excited oscillations. Hysteresis boundaries were derived in terms of the parameters \( \alpha, k_1, \) and \( k_2 \). The stability of periodic oscillations was analyzed, and the regions of unstable oscillations were separated from those of stable oscillations. The chaotic behavior of the model was also analyzed numerically, and it was found that chaos can arise through torus breakdown transition route, period-adding scenario, period-doubling, and sudden transition phenomena. Analog simulation was carried out to investigate routes to chaos. Comparing the results from different methods, we found a good agreement with explanations to small divergences.

An interesting question under investigation is that of proposing an analytical technique to derive a quasiperiodically forced Rayleigh equation in the case when the damping coefficient \( \beta_1 \) is large. It would be of interest to derive the hysteresis boundaries (in the forced Rayleigh oscillator) in terms of \( k_2 \) and out of the exact resonant point \( (\sigma \not= 0) \). Another problem under consideration is that of investigating the dynamics of a system consisting of a Rayleigh oscillator strongly coupled to an anharmonic oscillator of the Duffing type. An example of a model with practical interest that this coupled system can describe is an electrically loudspeaker, which is a device used in engineering and information technology.

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References